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1997 J. Phys. A: Math. Gen. 30 5997

(http://iopscience.iop.org/0305-4470/30/17/011)

J. Phys. A: Math. Gen. **30** (1997) 5997–6007. Printed in the UK PII: S0305-4470(97)80826-8

A covariant BRST approach of self-dual p-forms without extrafields

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Received 10 January 1997, in final form 14 April 1997

Abstract. Self-dual p-forms like first-order systems are investigated at classical, as well as at path integral level. Converting the self-dual system into a second-order gauge theory without introducing extrafields, we subsequently prove that: (i) the gauge theory, massive abelian p-form gauge fields and self-dual p-forms describe the same dynamics on the stationary surface of the field equations for the last model; (ii) self-dual p-forms and massive abelian p-form gauge fields represent a first-, respectively second-order BRST gauge-fixed version of the gauge system. The connection with the case of introducing extrafields is briefly addressed.

1. Introduction

Recently, the BRST formalism [1–5] has been successfully applied to second-class constrained systems by converting the original theory into a first-class one in different ways [6–9]. These conversion methods have been extended (with and without introducing extrafields) to a special type of second-class theories, such as massive abelian p-form gauge fields, which preserve in a certain fashion the reducibility relic of a gauge system [10, 11]. The importance of abelian p-form gauge fields is that they are profoundly connected with string theory and various supergravity models [12–13]. At the same time, from the mathematical point of view, on the one hand these objects allow us to understand zero modes and define some topological invariants associated with vector bundles (called characteristic classes) and, on the other hand, the differential forms are the only fields for which it is possible to define a differential operator independent of metric choice. A special class of models involving abelian p-form gauge fields is given by self-dual models. These models are described by the Lagrangian action

$$S_0^{\mathrm{L}}[A_{\mu_1\ldots\mu_p}] = \int \mathrm{d}^{2p+1}x\left(-\alpha\varepsilon_{\mu_1\ldots\mu_{2p+1}}F^{\mu_1\ldots\mu_{p+1}}A^{\mu_{p+2}\ldots\mu_{2p+1}} - \frac{M^2}{2\cdot p!}A_{\mu_1\ldots\mu_p}A^{\mu_1\ldots\mu_p}\right) = \int \mathrm{d}^{2p+1}x\,\mathcal{L}_0 \tag{1}$$

with $F^{\mu_1\ldots\mu_{p+1}} = \partial^{[\mu_1}A^{\mu_2\ldots\mu_{p+1}]}$, $\varepsilon_{\mu_1\ldots\mu_{2p+1}}$—the completely antisymmetric symbol in $(2p+1)$ dimensions and α a constant. The corresponding field equations read

$$\frac{\delta\mathcal{L}_0}{\delta A^{\mu_1\ldots\mu_p}} \equiv -\frac{M^2}{p!}A_{\mu_1\ldots\mu_p} - 2\alpha\varepsilon_{\nu_1\ldots\nu_{p+1}\mu_1\ldots\mu_p}F^{\nu_1\ldots\nu_{p+1}} = 0. \tag{2}$$

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We denote the stationary surface of field equations, (2), by Σ. Initially we specialize to the case p odd which is relevant by virtue of the subsequent analysis, the opposite situation being briefly addressed at the end of the paper.

The canonical analysis of (1) outputs the primary constraints

$$G_{i_1\ldots i_{p-1}} \equiv \pi_{0i_1\ldots i_{p-1}} = 0 \tag{3}$$

$$G_{i_1\ldots i_p} \equiv \pi_{i_1\ldots i_p} + \alpha(p+1)\varepsilon_{0i_1\ldots i_p j_1\ldots j_p} A^{j_1\ldots j_p} = 0 \tag{4}$$

the canonical Hamiltonian

$$H = \int \mathrm{d}^{2p}x \left(-2pA^{0i_2\ldots i_p}\partial^{i_1}\pi_{i_1\ldots i_p} + \frac{M^2}{2\cdot p!}A_{\mu_1\ldots\mu_p}A^{\mu_1\ldots\mu_p}\right) \tag{5}$$

and the secondary constraints

$$C_{i_2\ldots i_p} \equiv 2p\partial^{i_1}\pi_{i_1\ldots i_p} - \frac{M^2}{(p-1)!}A_{0i_2\ldots i_p} = 0. \tag{6}$$

It is obvious that the previous constraints are all second class, so action (1) has no gauge invariances. We mention that constraints (4) are a direct consequence of the first-order piece in the original action, while (3) and (6) are not linked to this feature (such constraints appear, for example, to massive abelian p-form gauge fields described by quadratic actions).

Starting from (1), we derive a gauge theory without introducing extrafields along the following steps: (i) we construct a system having only the primary constraints of the original one and the Hamiltonian $\overline{H}$ fulfilling $[\overline{H}, G_{i_1\ldots i_{p-1}}] = 0$ strongly; (ii) starting from the above system, we build a new one possessing only the primary constraints (3) and the Hamiltonian, $H' = \overline{H} +$ 'a series in $G_{i_1\ldots i_p}$', satisfying $[H', G_{i_1\ldots i_p}] = 0$ strongly; (iii) with the last theory at hand, we arrive at the searched for gauge theory having the first-class constraints $G_{i_1\ldots i_{p-1}} = 0$, $\gamma_{i_1\ldots i_{p-1}} \equiv \partial^{i_p}G_{i_1\ldots i_p} = 0$ and the first-class Hamiltonian $H^* = H' - p\int \mathrm{d}^{2p}x\, A^{0i_1\ldots i_{p-1}}\gamma_{i_1\ldots i_{p-1}}$. A remarkable characteristic of the Lagrangian action of the first-class theory resides in its Lorentz covariant quadratic form. Under these considerations, we are able to summarize the main results of our paper. (a) We prove that at the classical level (field equation level) self-dual p-forms, massive abelian p-form gauge fields and the gauge theory mentioned at step (iii) are equivalent on Σ (describe the same dynamics on Σ). In this context, it is shown that the Lagrangian actions of the gauge theory and massive abelian p-form gauge fields represent the second-order gauge, respectively, second-order non-gauge versions of self-dual p-forms. (b) Next we quantize the gauge theory in the antifield BRST formalism and establish, using some appropriate gauge-fixing fermions, its relationship at the path integral level with the other two models mentioned above. In this light, massive abelian p-form gauge fields and self-dual p-forms represent a second-, respectively, first-order gauge-fixed version of the same gauge theory. We remark that the path integrals of the massive and self-dual theories are not equivalent. Thus, the results (a) and (b) offer a complete description of the correlation among the three theories.

The paper is organized in six sections. Section 2 is devoted to the building up of a Lorentz covariant second-order gauge theory associated to self-dual p-forms. In section 3, we focus on the classical correlation among the original model, massive abelian p-form gauge fields and the gauge theory derived in section 2. Section 4 investigates the path integral link among the above three systems. In section 5 we briefly address the case p even, while section 6 ends the paper with some concluding comments.

Related to the antifield BRST quantization, we follow the general lines from [5].

2. The construction of the gauge theory

Here, we derive a gauge theory associated with system (1) by implementing steps (i)–(iii) without extending the phase-space. There already exist attempts at building up gauge theories without introducing extrafields [10], but their corresponding Lagrangian actions are not Lorentz covariant. The following treatment solves this deficiency, restoring the manifest relativistic covariance in an elegant fashion.

Initially, we intend to construct a Hamiltonian, $\overline{H}$, with the property

$$[\overline{H}, G_{i_1\ldots i_{p-1}}] = 0 \text{ strongly} \tag{7}$$

starting with the canonical Hamiltonian of self-dual p-forms, (5). If we solve this problem, the consistency of the primary constraints (3) vanishes identically and implies no secondary ones. In this way, we eliminate the secondary constraints, (6), from the theory such that the resulting system will possess only the primary constraints (3), (4) and the canonical Hamiltonian $\overline{H}$. Applying theorem 1 from [9], we find after simple computation

$$\overline{H} = \int \mathrm{d}^{2p}x \left(-\frac{2p\cdot p!}{M^2}(\partial_{i_1}\pi^{i_1\ldots i_p})^2 + \frac{M^2}{2\cdot p!}(A_{i_1\ldots i_p})^2\right) \tag{8}$$

where $(A_{i_1\ldots i_p})^2 = A_{i_1\ldots i_p}A^{i_1\ldots i_p}$ and a similar notation for the first term. This completes the implementation of (i).

Next we approach (ii), which reduces to the drop-out of the constraints (4). This is realized by building up a Hamiltonian H' fulfilling

$$[H', G_{i_1\ldots i_p}] = 0 \text{ strongly.} \tag{9}$$

First let us verify that (4) are no longer constraints for the theory described by the Hamiltonian H' and the primary constraints (3). In this respect let us consider the Hamiltonian action

$$S_0^H = \int \mathrm{d}^{2p+1}x(\dot{A}_{\mu_1\ldots\mu_p}\pi^{\mu_1\ldots\mu_p} - \mathcal{H}' - u^{i_1\ldots i_{p-1}}G_{i_1\ldots i_{p-1}} - u^{i_1\ldots i_p}G_{i_1\ldots i_p}) \tag{10}$$

with $\mathcal{H}'$ the density of H' and the u's Lagrange multipliers of the corresponding constraints. The equations of motion stemming from (10) are inferred to be $\dot{F} = [F, H'] + u^{i_1\ldots i_{p-1}}[F, G_{i_1\ldots i_{p-1}}] + u^{i_1\ldots i_p}[F, G_{i_1\ldots i_p}]$. The consistency of (4) leads, by means of (9), to the equations $\varepsilon_{0i_1\ldots i_p j_1\ldots j_p}u^{j_1\ldots j_p} = 0$, with the solution $u^{j_1\ldots j_p} = 0$. Substituting back the last solution in (10), we conclude that this action describes a theory having only the primary constraints (3). Second, we identify the solution of (9). In this end, we represent H' as a sum between $\overline{H}$ and a series in $G_{i_1\ldots i_p}$'s

$$H' = \overline{H} + \mu^{i_1\ldots i_p}G_{i_1\ldots i_p} + \mu^{i_1\ldots i_p}_{j_1\ldots j_p}G_{i_1\ldots i_p}G^{j_1\ldots j_p} + \cdots \tag{11}$$

with unknown μ's. Introducing (11) in (9), we deduce

$$H' = \int \mathrm{d}^{2p}x \left(-\frac{2\alpha^2\cdot(p!)^2\cdot(p+1)!}{M^2}(F^{i_1\ldots i_{p+1}})^2 + a(\overline{G}^{i_1\ldots i_p})^2\right) \tag{12}$$

where $a = M^2/8\alpha^2\cdot p!\cdot((p+1)!)^2$ and $\overline{G}^{i_1\ldots i_p} \equiv \pi^{i_1\ldots i_p} - \alpha(p+1)\varepsilon^{0i_1\ldots i_p j_1\ldots j_p}A_{j_1\ldots j_p}$. At this moment, step (ii) is also accomplished.

Finally, we pass to step (iii). We are in the position to generate a gauge theory with the primary constraints (3) provided their consistencies yield some secondary constraints

$$\gamma_{i_1\ldots i_{p-1}} = 0 \tag{13}$$

such that the entire set is first class. It is necessary to establish the concrete form of $\gamma_{i_1\ldots i_{p-1}}$ allowed by the first-class request. Once we succeed in determining H' satisfying (9), it is quite natural to exploit this result in order to render the computation easier and take $\gamma_{i_1\ldots i_{p-1}}$ to be functions of $G_{i_1\ldots i_p}$. We try the simplest dependence, which is obviously linear

$$\gamma_{i_1\ldots i_{p-1}} = f^{j_1\ldots j_p}_{i_1\ldots i_{p-1}} G_{j_1\ldots j_p} \tag{14}$$

with unknown coefficients. The first-class character of (13) consequently implies

$$[\gamma_{i_1\ldots i_{p-1}}, \gamma_{j_1\ldots j_{p-1}}] = M^{k_1\ldots k_{p-1}}_{i_1\ldots i_{p-1} j_1\ldots j_{p-1}} \gamma_{k_1\ldots k_{p-1}} \tag{15}$$

for some functions M. With the help of (14), we can derive a suitable non-trivial solution of (15), for some field-independent coefficients, $f^{j_1\ldots j_p}_{i_1\ldots i_{p-1}}$. In this case, (15) is expressed by

$$[\gamma_{i_1\ldots i_{p-1}}, \gamma_{j_1\ldots j_{p-1}}] = -2\alpha(p+1)\varepsilon_{0k_1\ldots k_p l_1\ldots l_p} f^{k_1\ldots k_p}_{i_1\ldots i_{p-1}} f^{l_1\ldots l_p}_{j_1\ldots j_{p-1}}. \tag{16}$$

The right-hand side of (16) clearly cannot reproduce the γ's, but can be taken to vanish choosing

$$f^{i_1\ldots i_p}_{j_1\ldots j_{p-1}} \equiv Z^{i_1\ldots i_p}_{j_1\ldots j_{p-1}} = \frac{1}{(p-1)!} \delta^{[i_1}_{j_1} \ldots \delta^{i_{p-1}}_{j_{p-1}} \partial^{i_p]} \tag{17}$$

so that (13) become abelian. Replacing (17) in (14), we determine the secondary constraints of the gauge theory under the form

$$\gamma_{i_1\ldots i_{p-1}} \equiv \partial^{i_p} G_{i_1\ldots i_p} = 0. \tag{18}$$

It is noteworthy that the above constraints are $(p-1)$-order reducible. Now, since $[H', G_{i_1\ldots i_{p-1}}] = 0$, it is clear that the first-class Hamiltonian of the searched for gauge theory can be taken as $H^* = H' - p\int \mathrm{d}^{2p}x\, A^{0i_1\ldots i_{p-1}} \gamma_{i_1\ldots i_{p-1}}$, so that $[H^*, G_{i_1\ldots i_{p-1}}] = -p\gamma_{i_1\ldots i_{p-1}} = 0$. The constant p is chosen for future convenience. The consistency of the last constraints induces no others because of (9) and the abelianity of the γ's. This solves step (iii).

In conclusion the resulting gauge theory is pictured by the extended action

$$S^E_0 = \int \mathrm{d}^{2p+1}x\, (\dot{A}_{\mu_1\ldots\mu_p} \pi^{\mu_1\ldots\mu_p} - \mathcal{H}^* - u^{i_1\ldots i_{p-1}} G_{i_1\ldots i_{p-1}} - v^{i_1\ldots i_{p-1}} \gamma_{i_1\ldots i_{p-1}}) \tag{19}$$

which is invariant under the gauge transformations

$$\delta_\varepsilon A^{0i_1\ldots i_{p-1}} = \varepsilon_1^{i_1\ldots i_{p-1}} \qquad \delta_\varepsilon A^{i_1\ldots i_p} = -\partial^{[i_1} \varepsilon_2^{i_2\ldots i_p]} \tag{20}$$

$$\delta_\varepsilon \pi_{0i_1\ldots i_{p-1}} = 0 \qquad \delta_\varepsilon \pi_{i_1\ldots i_p} = -\alpha p\,(p+1)\, \varepsilon_{0i_1\ldots i_p j_1\ldots j_p} \partial^{j_1} \varepsilon_2^{j_2\ldots j_p} \tag{21}$$

$$\delta_\varepsilon u^{i_1\ldots i_{p-1}} = \dot{\varepsilon}_1^{i_1\ldots i_{p-1}} \qquad \delta_\varepsilon v^{i_1\ldots i_{p-1}} = \dot{\varepsilon}_2^{i_1\ldots i_{p-1}} + \varepsilon_1^{i_1\ldots i_{p-1}} + \partial^{[i_{p-1}} \tilde{\varepsilon}^{i_1\ldots i_{p-2}]}. \tag{22}$$

The gauge parameters $\tilde{\varepsilon}$ appear due to the $(p-1)$-order reducibility of the secondary constraints, the kth order reducibility functions being $Z^{i_1\ldots i_{p-k}}_{j_1\ldots j_{p-k-1}} = (1/(p-k-1)!)\, \delta^{[i_1}_{j_1} \ldots \delta^{i_{p-k-1}}_{j_{p-k-1}} \partial^{i_{p-k}]}$. The Lagrangian action corresponding to (19) takes the form

$$S'^L_0[A_{\mu_1\ldots\mu_p}] = \int \mathrm{d}^{2p+1}x \left(\frac{2\alpha^2 \cdot (p!)^2 \cdot (p+1)!}{M^2} (F_{\mu_1\ldots\mu_{p+1}})^2 + \alpha \varepsilon_{\mu_1\ldots\mu_{2p+1}} F^{\mu_1\ldots\mu_{p+1}} A^{\mu_{p+2}\ldots\mu_{2p+1}} \right) = \int \mathrm{d}^{2p+1}x\, \mathcal{L}'_0 \tag{23}$$

and is invariant under the $(p-1)$-order reducible gauge transformations $\delta_\varepsilon A_{\mu_1\ldots\mu_p} = \partial_{[\mu_1} \varepsilon_{\mu_2\ldots\mu_p]}$. Formula (23) represents the cornerstone of this section. As underlined before, the Lagrangian action of the gauge theory is manifestly covariant and truly second-order.

This is implied by the elimination from the theory of the primary constraints (4). Indeed, this elimination produces a Hamiltonian H' quadratic in the momenta, and it is exactly this quadratic character which induces the same feature at the level of the corresponding Lagrangian action. It is clear that if (4) occurs, the Hamiltonian H^* reduces to $\overline{H}$ which, in turn, gives birth to a Lagrangian first-order theory.

3. The classical approach

In this section we investigate the classical correlation among self-dual p-forms, the gauge theory built previously and the massive abelian p-form gauge fields at both Lagrangian and Hamiltonian levels. As it will be seen massive abelian p-form gauge fields arise naturally in our analysis.

At the Lagrangian level, self-dual p-forms are described by action (1), with the corresponding stationary surface of field equations, Σ, expressed by (2). The Lagrangian action of the gauge theory is given by (23), the resulting field equations having the form

$$\frac{\delta \mathcal{L}'_0}{\delta A^{\nu_1 \ldots \nu_p}} \equiv 2\alpha \varepsilon_{\mu_1 \ldots \mu_{p+1} \nu_1 \ldots \nu_p} F^{\mu_1 \ldots \mu_{p+1}} - \frac{4\alpha^2 \cdot p! \cdot ((p+1)!)^2}{M^2} \partial^\nu F_{\nu \nu_1 \ldots \nu_p} = 0. \tag{24}$$

Then the relations between the functional derivatives implied in (2) and (24) read as

$$\frac{\delta \mathcal{L}'_0}{\delta A_{\mu_1 \ldots \mu_p}} = \frac{2\alpha \cdot (p+1)!}{M^2} \varepsilon^{\mu \mu_1 \ldots \mu_p \nu_1 \ldots \nu_p} \partial_\mu \frac{\delta \mathcal{L}_0}{\delta A^{\nu_1 \ldots \nu_p}}. \tag{25}$$

From (25) we can conclude that actions (1) and (23) describe the same dynamics on Σ as any solution of (2) is also a solution of (24). However, it is not true that any solution of (24) is a solution of (2), the class of solutions for (24) being larger than Σ. Indeed, if $A_{\mu_1 \ldots \mu_p} \in \Sigma$, then $A_{\mu_1 \ldots \mu_p} + \bar{\varepsilon}_{\mu_1 \ldots \mu_p} + \partial_{[\mu_1} \varepsilon_{\mu_2 \ldots \mu_p]}$, with $\bar{\varepsilon}_{\mu_1 \ldots \mu_p}$ some constants and $\varepsilon_{\mu_2 \ldots \mu_p}$ arbitrary functions, verifies (24), but not (2).

Because (24) and (2) must be compatible on Σ, we can replace the first term from (24) with the correspondent expression yielded by (2), obtaining the field equations

$$-\frac{M^2}{p!} A_{\nu_1 \ldots \nu_p} - \frac{4\alpha^2 \cdot p! \cdot ((p+1)!)^2}{M^2} \partial^\nu F_{\nu \nu_1 \ldots \nu_p} = 0. \tag{26}$$

These equations are nothing but the field equations of massive abelian p-form gauge fields and can be stemmed from the Lagrangian action

$$S''^{L}_0 = \int \mathrm{d}^{2p+1} x \left(\frac{2\alpha^2 \cdot (p!)^2 \cdot (p+1)!}{M^2} (F_{\mu_1 \ldots \mu_{p+1}})^2 - \frac{M^2}{2 \cdot p!} (A_{\mu_1 \ldots \mu_p})^2 \right) \equiv \int \mathrm{d}^{2p+1} x \, \mathcal{L}''_0. \tag{27}$$

From (2) and (26), we find

$$\frac{\delta \mathcal{L}''_0}{\delta A_{\mu_1 \ldots \mu_p}} = \left(\frac{2\alpha \cdot (p+1)!}{M^2} \varepsilon^{\mu \mu_1 \ldots \mu_p \nu_1 \ldots \nu_p} \partial_\mu + g^{\mu_1 \nu_1} \ldots g^{\mu_p \nu_p} \right) \frac{\delta \mathcal{L}_0}{\delta A^{\nu_1 \ldots \nu_p}} \tag{28}$$

where $g^{\mu\nu}$ denote the $(2p+1)$ spacetime metric. On behalf of (25) and (28), we conclude that any function belonging to Σ is simultaneously a solution of (2), (24) and (26), hence self-dual p-forms, the gauge theory and massive abelian p-form gauge fields display the same dynamics on the stationary surface, in other words they are equivalent on Σ.

Moreover, we can obtain the three previous actions from one another by substituting alternatively $F_{\mu_1 \ldots \mu_{p+1}}$ in terms of $A_{\mu_1 \ldots \mu_p}$ or conversely with the aid of (2). For example, beginning with the Lagrangian action of self-dual p-forms, (1), and replacing the fields $A^{\mu_1 \ldots \mu_p}$ from the first-order kinetic term with the expressions $-(2\alpha \cdot$

$p!/M^2)\varepsilon^{\nu_1\ldots\nu_{p+1}\mu_1\ldots\mu_p}F_{\nu_1\ldots\nu_{p+1}}$ resulting from (2), we derive precisely the Lagrangian action of massive abelian p-form gauge fields, (27). Thus, we remark that massive abelian p-form gauge fields represent the second-order non-gauge version of self-dual p-forms. The passing from the massive theory to the gauge system, (23), is acheived in a similar fashion, inserting a single $A_{\mu_1\ldots\mu_p}$ from the mass-term of (27) as a function of $F_{\nu_1\ldots\nu_{p+1}}$. The direct connection between self-dual p-forms and the gauge theory can also be shown by simultaneously performing the same operation in the kinetic and mass terms from (1). This means that the gauge system may be interpreted like a second-order gauge version of both the self-dual and massive theories. At the same time it is possible to follow the same lines as before, but in reverse order, i.e. we start with (23), employ (27) and consequently reach (1). Using the Lagrangian gauge conditions (2) we can conclude that the massive, as well as self-dual theory, represents a second-, respectively, first-order gauge-fixed version of the gauge system. Although equivalent on Σ, the three theories are not interchangeable because they do not describe the same physical phenomena.

At this point we can state the consequences of the equivalence on Σ among the three previous theories. The gauge variation of both sides in (25) is equal to zero, while the gauge variation of both sides in (28) is non-vanishing, being equal to the gauge variations of actions (1) and (27). In fact these considerations emphasize the gauge invariance of (23) and also the gauge non-invariance of (1) and (27). Inserting (25) in (28), we get

$$\frac{\delta\mathcal{L}_0''}{\delta A_{\mu_1\ldots\mu_p}} = \frac{\delta\mathcal{L}_0'}{\delta A_{\mu_1\ldots\mu_p}} + \frac{\delta\mathcal{L}_0}{\delta A_{\mu_1\ldots\mu_p}}. \tag{29}$$

Relations (29) show that

$$S_0'^{\mathrm{L}}[A_{\mu_1\ldots\mu_p}] = S_0''^{\mathrm{L}}[A_{\mu_1\ldots\mu_p}] - S_0^{\mathrm{L}}[A_{\mu_1\ldots\mu_p}]. \tag{30}$$

Formula (30) is the basic consequence of the above mentioned equivalence on Σ and expresses the fact that $-S_0''^{\mathrm{L}}$ plays the role of the Wess–Zumino action [14] for the self-dual theory and conversely, $-S_0^{\mathrm{L}}$ plays the same role for the massive model. Indeed, the massive and self-dual actions have the same gauge variations, which compensate through (30).

The link among these models within the classical Hamiltonian background is revealed by the correct canonical gauge conditions needed to be imposed to the gauge system in order to recover the massive or the self-dual theory. At the constraint level, the gauge system and massive abelian p-form gauge fields differ by the secondary constraints (18), respectively (6). Thus, the passing from the former to the latter theory is realized through the canonical gauge conditions $v^{i_1\ldots i_{p-1}} = 0$ and (6), where $v^{i_1\ldots i_{p-1}}$ are the Lagrange multipliers appearing in (19). On the other hand, self-dual p-forms have the primary constraints (4) in addition to those of massive abelian p-form gauge fields. Then, the passing from the gauge system to the self-dual theory is accomplished by supplementing the prior canonical gauge conditions with (4). The differences among the canonical gauge conditions in the Hamiltonian, respectively, Lagrangian forms, appear as a result of the different (for each of the second-class theories) use of (2). In terms of the canonical momenta of the gauge theory, equations (2) are turned into

$$-\frac{M^2}{p!}A_{0i_1\ldots i_{p-1}} - 2\alpha\varepsilon_{0i_1\ldots i_{p-1}j_1\ldots j_{p+1}}F^{j_1\ldots j_{p+1}} = 0 \tag{31}$$

$$-A_{i_1\ldots i_p} - \frac{1}{\alpha\cdot p!\cdot(p+1)!}\varepsilon_{0j_1\ldots j_p i_1\ldots i_p}\pi^{j_1\ldots j_p} = 0 \tag{32}$$

so the above equations describe Σ in the Hamiltonian approach. All the three theories have the (primary) constraints (3) in common. Exploiting (31) and (32) in an adequate manner, we reach the secondary constraints of massive and self-dual models. Indeed, by introducing (32) in (31) we find exactly (6), arriving in this way at massive abelian p-form gauge fields. The previous substitution is permissible because action (27) is quadratic and cannot produce constraints of the type (32), which are proportional to (4). Reprising the same procedure and using the constraints inferred by substituting (32) in (31), and also (32), we find the constraints of self-dual p-forms. The above discussion provides evidence that the election of (6), respectively (4) and (6), as canonical gauge conditions for the gauge system in order to infer the massive, respectively, self-dual theory is legitimate. This shows that the canonical correlation among the three models is based again on (the Hamiltonian correspondent of) Σ. It is exactly the prior canonical gauge conditions that will be implemented during the quantization process in order to emphasize the path integral connection among these systems. The above analysis clearly shows that the self-dual model possesses fewer physical degrees of freedom than the massive theory due to the greater number of second-class constraints associated with the former.

In this way the complete link among the three models at both the Lagrangian and Hamiltonian levels is elucidated, this connection constituting a basic result of this section and actually of this work.

4. The path integral approach

In the sequel we envisage establishing that massive abelian p-form gauge fields and self-dual p-forms are gauge-fixed versions of the gauge theory built earlier. To this end we quantize the extended action (19) of the gauge theory in the antifield BRST framework. The non-minimal solution of the master equation [5] reads as

$$\begin{aligned} S^E_{nm} = S^E_0 + \int \mathrm{d}^{2p+1}x \Bigg(& A^*_{0i_2\ldots i_p}\eta_1^{i_2\ldots i_p} - A^*_{i_1\ldots i_p}\partial^{[i_1}\eta_2^{i_2\ldots i_p]} \\ & -\alpha p(p+1)\varepsilon_{0i_1\ldots i_p j_1\ldots j_p}\pi^{*i_1\ldots i_p}\partial^{j_1}\eta_2^{j_2\ldots j_p} + u^*_{i_2\ldots i_p}\dot{\eta}_1^{i_2\ldots i_p} \\ & +v^*_{i_2\ldots i_p}(\dot{\eta}_2^{i_2\ldots i_p} + \eta_1^{i_2\ldots i_p} + \partial^{[i_p}\tilde{\eta}^{i_2\ldots i_{p-1}]}) + \bar{\eta}^*_{i_1\ldots i_p}B^{i_1\ldots i_p} \\ & +\sum_{k=0}^{p-2}\eta^*_{2i_1\ldots i_{p-k-1}}Z^{i_1\ldots i_{p-k-1}}_{j_1\ldots j_{p-k-2}}\eta_2^{j_1\ldots j_{p-k-2}} + \bar{\eta}^*_{1i_2\ldots i_p}B_1^{i_2\ldots i_p} + \bar{\eta}^*_{2i_2\ldots i_p}B_2^{i_2\ldots i_p} \\ & +\sum_{l=1}^{p-2}\overset{(l)}{\tilde{\eta}}{}^*_{i_1\ldots i_{p-l-2}}\overset{(l)}{\tilde{B}}{}^{i_1\ldots i_{p-l-2}} + \sum_{k=0}^{p-3}\sum_{l=1}^{p-k-2}\overset{(l)}{\bar{\eta}}{}^*_{i_1\ldots i_{p-k-l-2}}\overset{(l)}{B}{}^{i_1\ldots i_{p-k-l-2}}\Bigg). \end{aligned} \tag{33}$$

In (33) the A^*'s, π^*'s, u^*'s, and v^*'s are the ghost number minus one and Grassmann parity one antifields of the corresponding fields, $(\eta_a^{i_2\ldots i_p})_{a=1,2}$ and $\tilde{\eta}^{i_2\ldots i_{p-1}}$ denote the minimal ghost number one and Grassmann parity one ghosts associated with the gauge parameters from (20)–(22), while $\eta_2^{j_1\ldots j_{p-k-2}}$ and $\eta^*_{2j_1\ldots j_{p-k-2}}$ account for the minimal ghosts of ghosts etc, respectively, the associated antifields due to the $(p-1)$-order reducibility. The remaining fields stand for cohomologically trivial pairs.

Initially we make the link with the path integral of self-dual p-form gauge fields. The gauge-fixing fermion imposing the canonical gauge conditions mentioned at the end of the previous section is expressed by

$$\Psi = \int \mathrm{d}^{2p+1}x \Bigg(\bar{\eta}_{i_1\ldots i_p}G^{i_1\ldots i_p} - \frac{(p-1)!}{M^2}\bar{\eta}_{1i_2\ldots i_p}C^{i_2\ldots i_p} + \tilde{\eta}_{i_1\ldots i_{p-2}}Z^{i_1\ldots i_{p-2}}_{j_1\ldots j_{p-3}}\overset{(1)}{\tilde{\eta}}{}^{j_1\ldots j_{p-3}}$$

$$+\sum_{l=1}^{p-3}\overset{(l)}{\tilde{\eta}}_{i_1\ldots i_{p-l-2}}Z^{i_1\ldots i_{p-l-2}}_{j_1\ldots j_{p-l-3}}\overset{(l+1)}{\tilde{\eta}}{}^{j_1\ldots j_{p-l-3}}$$
$$+\sum_{k=0}^{p-3}\eta_{2i_1\ldots i_{p-k-2}}Z^{i_1\ldots i_{p-k-2}}_{j_1\ldots j_{p-k-3}}\overset{(1)}{\bar{\eta}}{}^{j_1\ldots j_{p-k-3}}+\bar{\eta}_{2i_2\ldots i_p}v^{i_2\ldots i_p}$$
$$+\sum_{k=0}^{p-4}\sum_{l=1}^{p-k-3}\overset{(l)}{\bar{\eta}}_{i_1\ldots i_{p-k-l-2}}Z^{i_1\ldots i_{p-k-l-2}}_{j_1\ldots j_{p-k-l-3}}\overset{(l+1)}{\bar{\eta}}{}^{j_1\ldots j_{p-k-l-3}}\Bigg). \tag{34}$$

Eliminating in the usual manner the antifields from (33) with the help of (34) and integrating in the corresponding path integral over all the fields but the $A^{i_1\ldots i_p}$'s, we get

$$Z_\Psi=\int\mathcal{D}A^{i_1\ldots i_p}\exp\mathrm{i}S' \tag{35}$$

with S' given by

$$S'=\int \mathrm{d}^{2p+1}x\Bigg(-\alpha(p+1)\varepsilon_{0i_1\ldots i_p j_1\ldots j_p}\dot{A}^{i_1\ldots i_p}A^{j_1\ldots j_p}-\frac{M^2}{2\cdot p!}(A_{i_1\ldots i_p})^2$$
$$+\frac{2\alpha^2\cdot(p!)^2\cdot(p+1)!}{M^2}(F_{i_1\ldots i_{p+1}})^2\Bigg). \tag{36}$$

Formula (35) expresses the Hamiltonian path integral over independent variables for the gauge theory associated with the original model and, in fact, it coincides with that of self-dual p-forms, which therefore represent a first-order gauge-fixed version of the gauge theory.

Now, let us see the link with the path integral with massive abelian p-form gauge fields. We invoke the correlation between the momenta of gauge and massive systems

$$\pi_{i_1\ldots i_p}=\Pi_{i_1\ldots i_p}+\alpha(p+1)\varepsilon_{0i_1\ldots i_p j_1\ldots j_p}A^{j_1\ldots j_p} \tag{37}$$

where $\Pi_{i_1\ldots i_p}$ obviously denote the massive theory momenta. As we also intend to use the solution (33) of the master equation in order to establish the link between the gauge and massive systems, we replace (37) in the prior solution and eliminate its topological character (the coupling through $\varepsilon_{0i_1\ldots i_p j_1\ldots j_p}$). This can be attained by inserting (37) in (31) and (32) and further substituting the resulting relations in (33). Thus the new solution stems from (33) by performing the above changes and discarding the non-minimal term $\bar{\eta}^*_{i_1\ldots i_p}B^{i_1\ldots i_p}$ which is no longer necessary as it implements the gauge conditions (4) which are now absent. Consequently the new gauge-fixing fermion will be found from (34) removing the first term and rewriting the functions $C_{i_2\ldots i_p}$ with the aid of (37). Denoting this gauge-fixing fermion by K, and integrating in the associated path integral over all fields excepting $(A^{i_1\ldots i_p},\Pi_{i_1\ldots i_p})$, we deduce

$$Z_K=\int\mathcal{D}A^{i_1\ldots i_p}\,\mathcal{D}\Pi_{i_1\ldots i_p}\exp\mathrm{i}\tilde{S} \tag{38}$$

with $\tilde{S}$ expressed by

$$\tilde{S}=\int\mathrm{d}^{2p+1}x\Bigg(\dot{A}^{i_1\ldots i_p}\Pi_{i_1\ldots i_p}-a(\Pi_{i_1\ldots i_p})^2-\frac{M^2}{2\cdot p!}(A_{i_1\ldots i_p})^2$$
$$+\frac{2\alpha^2\cdot(p!)^2\cdot(p+1)!}{M^2}(F_{i_1\ldots i_{p+1}})^2+\frac{p\cdot p!}{2M^2}(\partial^{i_p}\Pi_{i_1\ldots i_p})^2\Bigg). \tag{39}$$

The last formulae state that Z_K signifies the path integral over independent variables of massive abelian p-form gauge fields [10]. In this manner we have succeeded in showing that the massive theory represents a gauge-fixed version of the gauge theory (23). Relations (35)–(36) and (38)–(39) account for the main results of this section, and, actually, of the present paper.

At this point it is important to notice that the massive and self-dual path integrals, given by (35) and (38), are not equivalent. This is because the two theories do not have the same number of propagating degrees of freedom at the level of original fields (i.e. physical degrees of freedom). In fact the self-dual model possesses half the number of physical degrees of freedom of the massive system. Under these considerations we observe that the path integral of the self-dual theory is nothing but Z_K supplementary restricted to the paths fulfilling (4) expressed in terms of $\Pi_{i_1\ldots i_p}$. This is not a surprise as the difference between the above gauge-fixing fermions, Ψ and K, implements exactly the canonical gauge conditions (4). Indeed, restraining (38) to the above mentioned paths, we find

$$Z_K|_{G_{i_1\ldots i_p}=0} = \int \mathcal{D}A^{i_1\ldots i_p} \exp \mathrm{i}\tilde{S}' \tag{40}$$

with $\tilde{S}'$ of the form

$$\tilde{S}' = \int \mathrm{d}^{2p+1}x \left(-\alpha'(p+1)\varepsilon_{0i_1\ldots i_p j_1\ldots j_p} \dot{A}^{i_1\ldots i_p} A^{j_1\ldots j_p} - \frac{M'^2}{2\cdot p!}(A_{i_1\ldots i_p})^2 + \frac{2\alpha'^2\cdot(p!)^2\cdot(p+1)!}{M'^2}(F_{i_1\ldots i_{p+1}})^2 \right) \tag{41}$$

which is nothing but (36) up to some insignificant modifications of the constants. In conclusion, the gauge-fixing fermions Ψ and K induce different numbers of physical degrees of freedom for the two gauge-fixed theories such that the corresponding path integrals are not equivalent.

We have managed to show that the path integrals of massive and self-dual theories stand for some gauge-fixed versions of the gauge system obtained with the aid of some canonical gauge conditions describing the Hamiltonian representation of Σ. However, the massive and self-dual path integrals are not equivalent as they go beyond models with different numbers of physical degrees of freedom. More precisely, the self-dual path integral is inferred from the massive one restricting the latter to the paths satisfying (4).

5. The case p even

We will briefly address the case of p even. In this situation we can put the primary constraints in a form similar to the case p odd, the secondary ones being given by $C'_{i_1\ldots i_{p-1}} \equiv -(M^2/(p-1)!)A_{0i_1\ldots i_{p-1}} = 0$, $C'_{i_1\ldots i_p} \equiv \pi_{i_1\ldots i_p} - \alpha(p+1)\varepsilon_{0i_1\ldots i_p j_1\ldots j_p} A^{j_1\ldots j_p} = 0$, and the canonical Hamiltonian being expressed by $\tilde{H} = \int \mathrm{d}^{2p}x (M^2/2\cdot p!)(A_{\mu_1\ldots\mu_p})^2$. The new constraints are second-class, too, the consequence of the first-order piece in the original action being (4) and $C'_{i_1\ldots i_p} = 0$. Each of these constraint sets are separately first class, so one may regard one set as first-class, and the other as corresponding gauge conditions, in contrast to the case p odd, where the constraints (4) did not allow this split. The guiding line for obtaining some first-class constraints from (4) in the odd case was represented by the relations $[G_{i_1\ldots i_p}, G_{j_1\ldots j_p}] = 2\alpha(p+1)\varepsilon_{0i_1\ldots i_p j_1\ldots j_p}$, so that $[\partial^{i_p}G_{i_1\ldots i_p}, \partial^{j_p}G_{j_1\ldots j_p}] = 0$. In the even case, the number of constraints due to the first-order character is double and, because they are 'covariantly' separated, we can take $\partial^{i_p}C'_{i_1\ldots i_p} = 0$ or $\partial^{i_p}G_{i_1\ldots i_p} = 0$ as first-class constraints for the gauge theory. The derivative feature of the last constraints is not a strict necessity, as in the odd case, but it is merely suggested by this one. For this reason, the even case is less relevant for our method than the odd one. $\overline{H}$ and H' can, respectively, be given by $\hat{H} = \int \mathrm{d}^{2p}x\, (M^2/2 \cdot p!)(A_{i_1 \ldots i_p})^2$, $\hat{H}' = 0$. We notice that $\hat{H}'$ is the solution of $[\hat{H}', G_{i_1 \ldots i_p}] = 0$ and $[\hat{H}', C'_{i_1 \ldots i_p}] = 0$. Imposing as secondary first-class constraints of the gauge theory $\gamma'_{i_1 \ldots i_{p-1}} \equiv \partial^{i_p} C'_{i_1 \ldots i_p} = 0$, H^* reads $\hat{H}^* = \int \mathrm{d}^{2p}x\, (-pA^{0 i_1 \ldots i_{p-1}} \gamma'_{i_1 \ldots i_{p-1}} + g)$, where g satisfies the equations $[G_{i_1 \ldots i_{p-1}}, g] = 0$, $[\gamma'_{i_1 \ldots i_{p-1}}, g] = 0$, and thus ensures the Lorentz covariance of the associated Lagrangian action. Finally, we find the same Lagrangian action as in the odd case, namely (23). Therefore, the results already exposed in section 3 are still valid. Further, the quantization approach follows a similar line to the odd case, the canonical gauge conditions being supplemented with $C'_{i_1 \ldots i_p} = 0$ during the switch from the gauge to the self-dual theory.

6. Comments

We have shown in a consistent fashion that self-dual p-forms, massive abelian p-form gauge fields and the gauge theory (23) are equivalent at the classical level on the stationary surface Σ, both second-class systems representing at the same time two non-equivalent gauge-fixed versions of the gauge system. A remarkable feature is that although without adding extrafields, the gauge theory is still Lorentz covariant. The unconstrained system arising as an intermediate step in our procedure plays a crucial role in the process of building up the gauge theory. On the one hand, this system ensures the second-order character of the gauge theory and establishes the basis of its Lorentz covariance. On the other hand, the functions $G_{i_1 \ldots i_p}$ expressed by (4) are Hamiltonian constants of motion for this system, and, moreover, are identified with the conserved charges of some currents incorporated within the gauge theory field equations (24) through the relations

$$j^{\mu}_{\mu_1 \ldots \mu_p} = \frac{2\alpha \cdot p! \cdot (p+1)}{M^2} \varepsilon^{\mu}_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_p} \frac{\delta \mathcal{L}_0}{\delta A_{\nu_1 \ldots \nu_p}}. \tag{42}$$

In this sense, the unconstrained system is paternal for the gauge theory. Although the field equations (24) are invariant under the rigid transformations $A_{\nu_1 \ldots \nu_p} \rightarrow A_{\nu_1 \ldots \nu_p} + \bar{\varepsilon}_{\nu_1 \ldots \nu_p}$, action (23) is not so, and consequently the above currents do not result from Noether's Theorem [15].

The usefulness of these currents is essentially twofold. First, equations $j^{\mu}_{\mu_1 \ldots \mu_p} = 0$ span Σ and second, action (23) can be rewritten in terms of them under the form

$$S'^{\mathrm{L}}_0[A_{\mu_1 \ldots \mu_p}] = S^{\mathrm{L}}_0[A_{\mu_1 \ldots \mu_p}] + \frac{a}{p+1} \int \mathrm{d}^{2p+1}x\, (j_{\mu_1 \ldots \mu_{p+1}})^2. \tag{43}$$

From (43) it is easy to see that the gauge theory reduces to the self-dual model if $j^{\mu}_{\mu_1 \ldots \mu_p} = 0$. In this way the second term in (43) has the significance of the Wess–Zumino action [14] associated with (1).

The prior Wess–Zumino action helps us to make the connection with a different gauge theory involving extrafields associated with the original one. The gauge variation of the Wess–Zumino action is expressed by $\delta_\varepsilon (a/p+1)(j_{\mu_1 \ldots \mu_{p+1}})^2 = (M^2/p!)\partial_{[\mu_1} \varepsilon_{\mu_2 \ldots \mu_p]} A^{\mu_1 \ldots \mu_p}$. Introducing the extrafields $H_{\mu_1 \ldots \mu_{p-1}}$, with the gauge transformations $\delta_\varepsilon H_{\mu_1 \ldots \mu_{p-1}} = \partial_{[\mu_1} \varepsilon_{\mu_2 \ldots \mu_{p-1}]} + M\varepsilon_{\mu_1 \ldots \mu_{p-1}}$, we can substitute the above Wess–Zumino action with another one such that their gauge variations are equal. In this light, we make the transformation

$$\frac{a}{p+1} (j_{\mu_1 \ldots \mu_{p+1}})^2 = \frac{1}{p!} F^{\mu_1 \ldots \mu_p} \left(MA_{\mu_1 \ldots \mu_p} - \frac{1}{2} F_{\mu_1 \ldots \mu_p} \right) \tag{44}$$

where $F_{\mu_1\ldots\mu_p} = \partial_{[\mu_1} H_{\mu_2\ldots\mu_p]}$. Then, adding a gauge invariant term including p-forms, the Lagrangian action of the new gauge theory will be

$$\tilde{S}_0^{\prime\prime\mathrm{L}} = \int \mathrm{d}^{2p+1}x \left(-\alpha\varepsilon_{\mu_1\ldots\mu_{2p+1}} F^{\mu_1\ldots\mu_{p+1}} A^{\mu_{p+2}\ldots\mu_{2p+1}} - \frac{1}{2\cdot p!}(MA_{\mu_1\ldots\mu_p} - F_{\mu_1\ldots\mu_p})^2 + b(F_{\mu_1\ldots\mu_{p+1}})^2 \right) \tag{45}$$

with b a constant. The last gauge theory can be obtained by following a line mixing the procedure exposed in section 2 with that from [10], the constraints $\gamma_{i_1\ldots i_{p-1}} = 0$ being replaced with $-p\gamma_{i_1\ldots i_{p-1}} + Mp_{i_1\ldots i_{p-1}} = 0$, where $p_{i_1\ldots i_{p-1}}$ are the canonical momenta associated with $H^{i_1\ldots i_{p-1}}$.

As a final observation we mention that our analysis for $p = 1$ incorporates the results exposed in [16] investigating the equivalence between the self-dual vector model [17] and the topologically massive spin-one model [18].

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